JOURNAL OF APPROXIMATION THEORY 24, 330–343 (1978)

Near-Best L_1 Approximations on Circular and Elliptical Contours

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Communicated by E. W. Cheney

Received April 27, 1977

Polynomial approximations are obtained to analytic functions on circular and elliptical contours by forming partial sums of order n of their expansions in Taylor series and Chebyshev series of the second kind, respectively. It is proved that the resulting approximations converge in the L_1 norm as $n \to \infty$, and that they are near-best L_1 approximations within relative distances of the order of $\log n$. Practical implications of the results are discussed, and they are shown to provide a theoretical basis for polynomial approximation methods for the evaluation of indefinite integrals on contours.

1. INTRODUCTION

The problem of computing best polynomial approximations to functions on regions or contours in the complex plane is difficult and, except in the case of L_2 approximations, nonlinear. Although a simple and reliable algorithm has now been found [1] for best L_∞ approximation, it inevitably involves an iterative procedure. Linear projections, in contrast, may be computed directly and yet sometimes provide approximations very near to best. It is therefore important to study their properties.

In the case of the L_∞ norm it has already been proved that approximations of degree n near-minimax within relative distances of the order of $\log n$ may be obtained on circular contours by forming the partial sum of the Taylor series [2] and on elliptical contours by forming either the partial sum of the Chebyshev series of the first kind [3] or the interpolating polynomial at suitable Chebyshev nodes [4]. It is the purpose of the present paper to obtain comparable results in the L_1 norm.

The L_1 norm is of theoretical interest because of its dual relationship with L_∞, and it is also of practical relevance if we wish to smooth out a given function or approximate its integral. Polynomial approximations formed by projections in L_1 have already been studied for the real interval $[-1, 1]$ in [5],

0021-9045/78/0244–0330$02.00/0

where it was established that the partial sum of the Chebyshev series of the second kind is near-best in L_1 within a relative distance of the order of log n, and in [6] where the same approximation was proved to be convergent in L_1 as $n \to \infty$. We now generalize these two results to elliptical contours, obtaining results for L_1 and Chebyshev polynomials of the second kind closely analogous to those of [3] for L_∞ and Chebyshev polynomials of the first kind. We also study the Taylor series projection on a circular contour in L_1, and prove results analogous to those given in [2] for the same projection in L_∞.

Finally we show that these new results provide a theoretical basis for polynomial approximation methods for the evaluation of indefinite integrals on contours.

2. Projections on a Function Space

Consider a contour Γ, which is taken to be either a circle C_ρ of radius $\rho > 1$ centered at the origin, or the ellipse ξ_ρ with foci at ± 1 and semiaxes of lengths $\frac{1}{2}(\rho \pm \rho^{-1})$, where ρ is fixed. The ellipse is defined only for $\rho > 1$ and collapses to the real interval $[-1, 1]$ as ρ approaches 1. Denote the interior of Γ by $I(\Gamma)$ and the closure by $\overline{I(\Gamma)}$.

The function space $A(\Gamma)$ is defined to be the linear space of functions f which are continuous on $\overline{I(\Gamma)}$ and analytic in $I(\Gamma)$, normed by the L_1 norm on Γ

$$\|f\|_1 = \int_\Gamma |f(z)| \, |dz|.$$

Obviously this norm, which is measured over Γ only, is independent of values of f in $I(\Gamma)$, and so attention is restricted to problems in which approximations are only required on the contour.

The polynomial subspace Π_n is the space of complex algebraic polynomials of degree not exceeding n, and clearly a best approximation f_n^B exists in Π_n to any given f in $A(\Gamma)$.

A mapping $M: A(\Gamma) \to \Pi_n$ and a corresponding approximation Mf are said to be near-best within a relative distance μ if they satisfy the inequality

$$\|f - Mf\| \leqslant (1 + \mu) \cdot \|f - f_n^B\|.$$

In particular any projection P, which is defined to be a bounded linear idempotent mapping, is always near-best within a relative distance $\| P \|$, since it can easily be established (see [5]) that

$$\|f - Pf\| \leqslant (1 + \| P \|) \cdot \|f - f_n^B\|.$$

The norm of P is defined here as

$$\|P\| = \sup_{f\in A(\Gamma)} \frac{\|Pf\|}{\|f\|}. \tag{1}$$

If we can show that $\|P\|$ is "appropriately small," then Pf will be very close to a best approximation. For example, if $\|P\|$ does not exceed 9, then Pf is within one decimal place of the accuracy of f_n^B. In the two examples in the present paper, bounds for $\|P\|$ are obtained which behave like $\log n$ and only exceed 9 for extremely large values of n.

The mappings to which we shall restrict our attention are those projections which are formed by taking the partial sum of an expansion of f on $\overline{I(\Gamma)}$ in polynomials orthogonal with respect to a weight function W on Γ. Such polynomials $\{\phi_k(z)\}$, where ϕ_k is of degree k, are defined by

$$\int_\Gamma \phi_m(z)\,\overline{\phi_n(z)}\,\mathrm{W}(z)\,|dz| = 0 \qquad \text{for} \quad m \neq n. \tag{2}$$

In general we define the L_p norm of a function $f(z)$ on a contour Γ to be

$$\|f(z)\|_p = \left[\int_\Gamma |f(z)|^p\,|dz|\right]^{1/p}.$$

Before proceeding further, we establish an analog for the circular contour C_ρ: $|z| = \rho$ of a standard result for L_p norms on the real line.

LEMMA 2.1. *If a sequence of continuous approximations converges to a continuous function in the L_p norm on C_ρ, then it will converge in the L_q norm on C_ρ for any $q < p$.*

Proof. Suppose that continuous functions f and g are defined on C_ρ, and write

$$F(\theta) = |f(\rho e^{i\theta})|, \qquad G(\theta) = |g(\rho e^{i\theta})|.$$

Then, if $1/p + 1/p' = 1$, Hölder's inequality gives

$$\int_0^{2\pi} F(\theta)\,G(\theta)\,d\theta \leqslant \left[\int_0^{2\pi} |F(\theta)|^p\,d\theta\right]^{1/p}\left[\int_0^{2\pi} |G(\theta)|^{p'}\,d\theta\right]^{1/p'}.$$

Since $|dz| = \rho\,d\theta$, we deduce an analog of Hölder's inequality

$$\int_{C_\rho} |f(z)|\,|g(z)|\,|dz| \leqslant \left[\int_{C_\rho} |f(z)|^p\,|dz|\right]^{1/p}\left[\int_{C_\rho} |g(z)|^{p'}\,|dz|\right]^{1/p'}.$$

By taking $p = q/r$ for $q > r$, $f(z) = [\phi(z)]^r$, and $g(z) = 1$, we deduce in the usual way that

$$\| \phi(z)\|_r \leqslant \| \phi(z)\|_q \left[\int_{C_\rho} | dz |\right]^{1/r-1/q}$$

and the lemma follows by considering a sequence of functions.

3. The Taylor Series and the Circle

It is easily verified (see [7]) that the orthogonality relationship (2) is satisfied for the polynomials $\phi_k(z) = z^k$ on the contour $\Gamma = C_\rho$ with weight function $W(z) = 1$. Thus the Taylor series of f is an orthogonal expansion on Γ, and the mapping S_n of f on the partial sum of degree n of the series is a projection. We are now ready to prove the results we require.

Theorem 3.1. *$\{S_n f\}$ converges to f in the L_1 norm on C_ρ.*

Proof. $\{S_n f\}$ converges to f in the L_2 norm on C_ρ as a consequence of the orthogonality of $\{z^k\}$. The result follows by Lemma 2.1.

Theorem 3.2. *$\| S_n \|_1 \leqslant \tau_n$, where*

$$\tau_n = \frac{1}{\pi} \int_0^\pi \left| \frac{\sin(n+1)\theta}{\sin\theta} \right| d\theta.$$

Proof. By applying Cauchy's integral formulas it can be shown (see [2]) that

$$(S_n f)(z) = \frac{1}{2\pi} \int_{C_1} \frac{s^{n+1}-1}{s^{n+1}(s-1)} f(zs)\, ds.$$

Hence

$$\begin{aligned} \|(S_n f)(z)\|_1 &= \frac{1}{2\pi} \int_{C_\rho} \left| \int_{C_1} \frac{s^{n+1}-1}{s^{n+1}(s-1)} f(zs)\, ds \right| | dz | \\ &\leqslant \frac{1}{2\pi} \int_{C_\rho} \int_{C_1} \left| \frac{s^{n+1}-1}{s-1} \right| | f(zs) | \, | ds | \, | dz | \\ &= \frac{1}{2\pi} \int_{C_1} \left| \frac{s^{n+1}-1}{s-1} \right| \left\{ \int_{C_\rho} | f(zs) | \, | dz | \right\} | ds |, \end{aligned}$$

where the change in the order of integration is justified by Fubini's theorem. The transformation $t = zs$ now gives

$$\|(S_n f)(z)\|_1 \leqslant \frac{1}{2\pi} \int_{C_1} \left| \frac{s^{n+1}-1}{s-1} \right| | ds | \cdot \int_{C_\rho} | f(t) | \, | dt |.$$

By making the transformation $s = e^{i\theta}$ (see [2]) we find that

$$\frac{1}{2\pi} \int_{C_1} \left| \frac{s^{n+1} - 1}{s - 1} \right| |ds| = \tau_n .$$

Hence

$$\|(S_n f)(z)\|_1 \leqslant \tau_n \cdot \|f\|_1$$

and by (1)

$$\| S_n \|_1 \leqslant \tau_n . \qquad \text{Q.E.D.}$$

In [2] it was observed that

$$\tau_n \sim \frac{4}{\pi^2} \log n,$$

and thus $\| S_n \|_1$ has the required type of asymptotic behavior. We also note that τ_{2n} is equal to λ_n, the nth Lebesgue constant.

In [2] it was established that

$$\| S_n \|_\infty \leqslant \tau_n ,$$

and hence we have an identical bound in L_1 to that in L_∞.

4. The Chebyshev Series of the Second Kind and the Ellipse

The ellipse ξ_ρ in the w-plane with foci ± 1 and semiaxes $\frac{1}{2}(\rho \pm \rho^{-1})$ has the equation

$$| w + (w^2 - 1)^{1/2} | = \rho,$$

where $\rho > 1$. Hence the mapping

$$z = w + (w^2 - 1)^{1/2}, \tag{3}$$

which has the inverse mapping

$$w = \tfrac{1}{2}(z + z^{-1}), \tag{4}$$

takes $I(\xi_\rho)$ with $[-1, 1]$ deleted into the open annulus N_ρ: $1 < | z | < \rho$. We denote by N_ρ^* the larger annulus $\rho^{-1} < | z | < \rho$.

In terms of the complex variables w and z, the Chebyshev polynomials of degree n, $T_n(w)$ and $U_n(w)$, of the first and second kinds, respectively, may be defined by

$$T_n(w) = \tfrac{1}{2}(z^n + z^{-n}) \tag{5}$$

and

$$(w^2 - 1)^{1/2} \cdot U_{n-1}(w) = \tfrac{1}{2}(z^n - z^{-n}). \tag{6}$$

By making the transformation (4) and setting $z = \rho e^{i\theta}$, it is easy to verify that

$$\int_{\xi_\rho} U_{m-1}(w)\, \overline{U_{n-1}(w)}\, | w^2 - 1 |^{1/2}\, | dw |$$

is equal to

$$\tfrac{1}{4} \int_0^{2\pi} (\rho^m e^{im\theta} - \rho^{-m} e^{-im\theta})(\rho^n e^{-in\theta} - \rho^{-n} e^{in\theta})\, d\theta,$$

an integral which vanishes for $m \neq n$. Hence the polynomials $\{U_k(w)\}$ are orthogonal with respect to $| w^2 - 1 |^{1/2}$ on the ellipse ξ_ρ. Moreover, the mapping H_n of any function f in $A(\xi_\rho)$ on the partial sum of degree n of its expansion on $\overline{I(\xi_\rho)}$ in Chebyshev series of the second kind is a projection.

For $f(w)$ in $A(\xi_\rho)$, define the function

$$g(z) = (z - z^{-1}) f(\tfrac{1}{2}(z + z^{-1})), \tag{7}$$

where z and w are related by (3). Then

$$g(z) = 2(w^2 - 1)^{1/2} f(w) \qquad \text{in } N_\rho\,.$$

Now, by (7), g is analytic in N_ρ^* and continuous on $\overline{N_\rho^*}$. Hence applying Cauchy's integral formulas, g has the Laurent series expansion

$$g(z) = \sum_{k=-\infty}^{\infty} a_k z^k,$$

where

$$a_k = \frac{1}{2\pi i} \int_{C_\rho} \frac{g(z)}{z^{k+1}}\, dz. \tag{8}$$

By (7) it is clear that the coefficients satisfy

$$a_k = -a_{-k} \qquad \text{for all } k,$$

and in particular $a_0 = 0$. Hence

$$g(z) = \sum_{k=1}^{\infty} a_k (z^k - z^{-k}). \tag{9}$$

Now, for positive k,

$$a_{-k} = \frac{1}{2\pi i} \int_{C_\rho} \frac{g(z)}{z^{-k+1}}\, dz. \tag{10}$$

But

$$a_k = \tfrac{1}{2}(a_k - a_{-k}),$$

and hence from (8) and (10),

$$a_k = -\frac{1}{4\pi i}\int_{C_\rho} g(z)(z^k - z^{-k})\frac{dz}{z}. \tag{11}$$

Transforming to the w-plane and using the definition (6) of $U_{n-1}(w)$, (9), and (11) give

$$g(z) = 2(w^2 - 1)^{1/2} f(w) = \sum_{k=1}^{\infty} a_k\, 2(w^2 - 1)^{1/2}\, U_{k-1}(w)$$

where

$$a_k = -\frac{1}{4\pi i}\int_{\xi_\rho} 2(w^2 - 1)^{1/2} f(w)\cdot 2(w^2 - 1)^{1/2}\, U_{k-1}(w)\cdot\frac{dw}{(w^2 - 1)^{1/2}}.$$

Hence

$$f(w) = \sum_{k=1}^{\infty} a_k U_{k-1}(w), \tag{12}$$

where

$$a_k = -\frac{1}{\pi i}\int_{\xi_\rho} (t^2 - 1)^{1/2}\, U_{k-1}(t) f(t)\, dt. \tag{13}$$

Also

$$(H_n f)(w) = \sum_{k=1}^{n+1} a_k U_{k-1}(w) \tag{14}$$

for the same a_k.

Thus the expansion of $g(z)$ in Laurent series on $\bar{N}_\rho$ is exactly $2(w^2 - 1)^{1/2}$ times the expansion of $f(w)$ in Chebyshev series of the second kind on $\overline{I(\xi_\rho)}$. And precisely the same coefficients a_k occur in both expansions, when they are expressed in forms (9) and (12). We are now ready to prove the results we require.

THEOREM 4.1. *$\{H_n f\}$ converges to f in the L_1 norm on ξ_ρ.*

Proof. The Laurent series of $g(z)$, analytic in N_ρ, and continuous on $\bar{N}_\rho$, converges in the L_2 norm on the outside contour C_ρ as a consequence of the orthogonality of the positive and negative powers of z on C_ρ. Hence the series converges in the L_1 norm on C_ρ by Lemma 2.1, and

$$\lim_{n\to\infty}\int_{C_\rho}\left| g(z) - \sum_{k=1}^{n+1} a_k(z^k - z^{-k})\right| |dz| = 0.$$

Transforming to the w-plane,

$$\lim_{n\to\infty}\int_{\xi_\rho}\left| 2(w^2 - 1)^{1/2} f(w) - 2(w^2 - 1)^{1/2}\sum_{k=1}^{n+1} a_k U_{k-1}(w)\right| \frac{|dw|}{|w^2 - 1|^{1/2}} = 0,$$

since

$$dz = \frac{w + (w^2 - 1)^{1/2}}{(w^2 - 1)^{1/2}}\, dw \qquad \text{and} \qquad | w + (w^2 - 1)^{1/2} | = \rho.$$

Hence

$$\lim_{n\to\infty} \int_{\xi_\rho} \left| f(w) - \sum_{k=1}^{n+1} a_k U_{k-1}(w) \right| | dw | = 0,$$

and by (14) the result is proved.

THEOREM 4.2. $\| H_n \|_1 \leqslant \lambda_{n+1}$, *where* λ_n *is Lebesgue's constant*

$$\lambda_n = \frac{1}{\pi} \int_0^\pi \left| \frac{\sin(n + \frac{1}{2}\theta)}{\sin \frac{1}{2}\theta} \right| d\theta. \tag{15}$$

Proof. From (13) and (14),

$$(H_n f)(w) = -\frac{1}{\pi i} \int_{\xi_\rho} \left\{ \sum_{k=1}^{n+1} U_{k-1}(t)\, U_{k-1}(w) \right\} f(t)(t^2 - 1)^{1/2}\, dt.$$

Now

$$0 = \frac{1}{\pi i} \int_{\xi_\rho} \left\{ \sum_{k=1}^{n+1} T_k(t)\, T_k(w) \right\} f(t)(w^2 - 1)^{-1/2}\, dt,$$

since the integrand is analytic in $I(\xi_\rho)$. Hence, by addition,

$$(H_n f)(w) = \frac{1}{\pi i} \int_{\xi_\rho} \left\{ \sum_{k=1}^{n+1} [T_k(t)\, T_k(w) - U_{k-1}(t)(t^2 - 1)^{1/2}\, U_{k-1}(w)(w^2 - 1)^{1/2}] \right\}$$
$$\times \frac{f(t)}{(w^2 - 1)^{1/2}}\, dt.$$

Since

$$T_k(w) = \cos(k \cos^{-1} w)$$

and

$$U_{k-1}(w) = \frac{\sin(k \cos^{-1} w)}{\sin(\cos^{-1} w)} = \frac{-i \sin(k \cos^{-1} w)}{(w^2 - 1)^{1/2}},$$

we deduce that

$$(H_n f)(w) = \frac{1}{\pi i} \int_{\xi_\rho} \left\{ \sum_{k=1}^{n+1} \cos k(\cos^{-1} t - \cos^{-1} w) \right\} \frac{f(t)}{(w^2 - 1)^{1/2}}\, dt.$$

If the summation over k is extended to include $k = 0$, an analytic function is added to the integrand, and the integral is unaffected. Thus

$$(H_n f)(w) = \frac{1}{\pi i} \int_{\xi_\rho} \sum_{k=0}^{n+1}{}' \cos k\theta \, \frac{f(t)}{(w^2 - 1)^{1/2}} \, dt,$$

where

$$\theta = \cos^{-1} t - \cos^{-1} w$$

and the dash denotes that the first term in the summation is halved. Hence

$$\begin{aligned} \|(H_n f)(w)\|_1 &= \frac{1}{\pi} \int_{\xi_\rho} \left| \int_{\xi_\rho} \sum_{k=0}^{n+1}{}' \cos k\theta \, \frac{f(t)}{(w^2 - 1)^{1/2}} \, dt \right| \cdot | \, dw \, | \\ &\leqslant \frac{1}{\pi} \int_{\xi_\rho} \int_{\xi_\rho} \left| \sum_{k=0}^{n+1}{}' \cos k\theta \right| \cdot \frac{| f(t) |}{| w^2 - 1 |^{1/2}} \, | \, dt \, | \, | \, dw \, | \\ &= \frac{1}{\pi} \int_{\xi_\rho} | f(t) | \left\{ \int_{\xi_\rho} \left| \sum_{k=0}^{n+1}{}' \cos k\theta \right| \frac{| \, dw \, |}{| w^2 - 1 |^{1/2}} \right\} | \, dt \, |. \end{aligned}$$

Both t and w move on ξ_ρ, and hence both $\cos^{-1} t$ and $\cos^{-1} w$ are of the form

$$\alpha - i \log \rho \qquad \text{for some } \alpha \text{ in } [-\pi, \pi].$$

Thus θ takes real values, moving from $-\pi$ to π as w moves round ξ_ρ for fixed t. Transforming from w to θ,

$$\| H_n f \|_1 \leqslant \frac{1}{\pi} \int_{\xi_\rho} | f(t) | \left\{ \int_{-\pi}^{\pi} \left| \sum_{k=0}^{n+1}{}' \cos k\theta \right| d\theta \right\} | \, dt \, |,$$

since

$$dw = i(w^2 - 1)^{1/2} \, d\theta.$$

But, by a classical result in Fourier series,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{n}{}' \cos k\theta \right| d\theta = \frac{1}{\pi} \int_0^{\pi} \left| \frac{\sin(n + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta} \right| d\theta,$$

and hence by (15)

$$\| H_n f \|_1 \leqslant \lambda_{n+1} \| f \|_1 .$$

The result follows.

5. Implications in Approximation Theory

On the circle we have now established that the partial sum of the Taylor series is not only a best approximation in L_2, but also a near-best approximation in both L_1 and L_∞. Thus it is a powerful approximation in all three principal Hölder norms.

On the ellipse we now know that the Chebyshev polynomials of the second kind fulfill a role in near-best L_1 approximation similar to that played by the Chebyshev polynomials of the first kind in near-best L_∞ approximation. Moreover both kinds of Chebyshev polynomials are orthogonal with respect to appropriate weight functions. Thus the partial sums of the Chebyshev series of the first and second kinds provide between them powerful approximations in L_1, L_2 weighted by $|w^2-1|^{1/2}$ or $|w^2-1|^{-1/2}$, and L_∞.

6. Applications to Indefinite Integration

Suppose that a closed contour Γ passes through a fixed point z_0 and that z is a general point of Γ. Let Γ_z denote the portion of Γ between z_0 and z, and consider the problem of computing the indefinite integral

$$g(z) = \int_{\Gamma_z} f(z)\,dz \tag{16}$$

for all z on Γ, where f is a given function in $A(\Gamma)$.

If $f(z)$ is replaced by a polynomial approximation $f_n(z)$ of degree n on Γ, then $g(z)$ will be replaced by the polynomial

$$g_{n+1}(z) = \delta_n + \int_{\Gamma_z} f_n(z)\,dz \tag{17}$$

of degree $n+1$. A constant of integration δ_n, equal to $g_{n+1}(z_0)$, is included for generality. If we define

$$\| g(z) - g_{n+1}(z)\|_\infty = \sup_{z\in\Gamma} | g(z) - g_{n+1}(z)|,$$

then, by analysis similar to that adopted in [6],

$$\begin{aligned} \| g(z) - g_{n+1}(z)\|_\infty &= \left\| -\delta_n + \int_{\Gamma_z} (f-f_n)\,dz \right\|_\infty \\ &\leqslant |\delta_n| + \left\| \int_{\Gamma_z} (f-f_n)\,dz \right\|_\infty \\ &\leqslant |\delta_n| + \left\| \int_{\Gamma_z} |f-f_n|\,|dz| \right\|_\infty \\ &= |\delta_n| + \int_{\Gamma} |f-f_n|\,|dz|. \end{aligned}$$

Thus

$$\| g - g_{n+1} \|_\infty \leqslant |\delta_n| + \| f - f_n \|_1 . \tag{18}$$

By Theorems 3.1 and 4.1, we may immediately deduce the following two results from (18):

THEOREM 6.1. *On the circular contour C_ρ, if f_n is chosen to be the partial sum $S_n f$ of the Taylor series of f, and if g_{n+1} is defined by* (17) *for any δ_n such that $|\delta_n| \to 0$ as $n \to \infty$, then g_{n+1} converges uniformly to g as $n \to \infty$.*

THEOREM 6.2. *On the elliptical contour ξ_ρ, if f_n is chosen to be the partial sum $H_n f$ of the Chebyshev series of the second kind of f, and if g_{n+1} is defined by* (17) *for any δ_n such that $|\delta_n| \to 0$ as $n \to \infty$, then g_{n+1} converges uniformly to g as $n \to \infty$.*

Moreover, by Theorems 3.2 and 4.2, if f_n is chosen to be $S_n f$ on C_ρ and $H_n f$ on ξ_ρ, then the bound (18) on the maximum error in the indefinite integral will be within a relative distance of the order of $\log n$ of its minimum value.

However, since the bound (18) on $\| g - g_{n+1} \|_\infty$ is very pessimistic, it is not clear that its minimization will in fact provide a near-minimization of $\| g - g_{n+1} \|_\infty$. Nevertheless it does. For we can show (as in [6] for the real line) that the approximations $S_n f$ and $H_n f$ yield the minimax approximations g_{n+1} to g on their respective contours if f is a model function, namely a monic polynomial of degree $n + 1$. In this case, the error $f - f_n$ in the integrand of (16) is also a monic polynomial of degree $n + 1$, and its integral $g - g_{n+1}$ is a polynomial of degree $n + 2$ with fixed leading coefficient $(n + 2)^{-1}$.

By applying the characterization theorem for minimax polynomial approximations on contours, established by Rivlin and Shapiro [8], it can be verified that the zero function is the best nth degree polynomial approximation to the functions z^{n+1} and $2^{-n} T_{n+1}(w)$ on C_ρ and ξ_ρ, respectively (where we use the variable w on ξ_ρ). The relevant extremal points in these characterizations are

$$z = z_k = \rho e^{ik\pi/(n+1)}, \qquad k = 0, 1, \ldots, 2n + 1 \text{ on } C_\rho ,$$

and

$$w = w_k = \tfrac{1}{2}(z_k + z_k^{-1}), \qquad k = 0, 1, \ldots, 2n + 1 \text{ on } \xi_\rho .$$

Suitable extremal signatures are, respectively, the complex signs at the extremal points of the functions

$$\mu(z) = z^{n+1} \qquad \text{on } C_\rho$$

and

$$\mu(w) = T_{n+1}(w) = \tfrac{1}{2}(z^{n+1} + z^{-n-1}) \qquad \text{on } \xi_\rho ,$$

where

$$w = \tfrac{1}{2}(z + z^{-1}).$$

Thus $\| g - g_{n+1} \|_\infty$ is minimized for the model function if

$$g(z) - g_{n+1}(z) = (n + 2)^{-1} z^{n+2} \qquad \text{on } C_\rho$$

and

$$g(z) - g_{n+1}(z) = (n + 2)^{-1} 2^{-1-n} T_{n+2}(z) \qquad \text{on } \xi_\rho .$$

(We now revert to z variables on both contours.)

Since $g(z) - g_{n+1}(z) = -\delta_n$ when $z = z_0$, the corresponding choices of constants of integration are

$$\delta_n = -(n + 2)^{-1} z_0^{n+2} \qquad \text{on } C_\rho$$

and

$$\delta_n = -(n + 2)^{-1} 2^{-1-n} T_{n+2}(z_0) \qquad \text{on } \xi_\rho .$$

But

$$\frac{d}{dz} [(n + 2)^{-1} z^{n+2}] = z^{n+1}$$

and

$$\frac{d}{dz} [(n + 2)^{-1} 2^{-1-n} T_{n+2}(z)] = 2^{-1-n} U_{n+1}(z).$$

Thus

$$f(z) - f_n(z) = z^{n+1} \qquad \text{on } C_\rho$$

and

$$f(z) - f_n(z) = 2^{-1-n} U_{n+1}(z) \qquad \text{on } \xi_\rho ,$$

both of which are monic polynomials of degree $n + 1$. Hence $\| g - g_{n+1} \|_\infty$ is minimized when we choose $S_n f$ for f_n on C_ρ and $H_n f$ for f_n on ξ_ρ, and we have the following results:

THEOREM 6.3. *On C_ρ, the minimax polynomial approximation to the indefinite integral from z_0 to z of the model function $f(z) = z^{n+1}$ is obtained by integrating the partial sum $f_n(z) = (S_n f)(z)$ of the Taylor series of f and choosing as constant of integration $\delta_n = -(n + 2)^{-1} z_0^{n+2}$.*

THEOREM 6.4. *On ξ_ρ, the minimax polynomial approximation to the indefinite integral from z_0 to z of the model function $f(z) = z^{n+1}$ is obtained by* *integrating the partial sum $f_n(z) = (H_n f)(z)$ of the Chebyshev series of the second kind of f and choosing as constant of integration*

$$\delta_n = -(n + 2)^{-1} 2^{-1-n} T_{n+2}(z_0).$$

If δ_n is chosen to be zero for the model function, the approximation obtained to g is still very close to minimax. For the error $g - g_{n+1}$ is then

$$(n + 2)^{-1}(z^{n+2} - z_0^{n+2}) \qquad \text{on } C_\rho \tag{19}$$

and

$$(n + 2)^{-1} 2^{-1-n}(T_{n+2}(z) - T_{n+2}(z_0)) \qquad \text{on } \xi_\rho . \tag{20}$$

The polynomial (19) has $n + 2$ equally spaced zeros on the contour C_ρ, and (20) has $n + 2$ corresponding zeros on ξ_ρ. These respective zeros are in fact precisely the appropriate nodes for near-minimax interpolation on C_ρ (see [2]) and on ξ_ρ (see [4]).

From the above results for the model function, we expect that in the case of a general function f in $A(\Gamma)$ the approximations obtained by integrating $S_n f$ and $H_n f$ on the respective contours are near-minimax.

A simpler process in practice than the determination of $S_n f$ and $H_n f$ is to form f_n from f by interpolating on C_ρ and ξ_ρ at an appropriate set of $n + 1$ points, such as

$$\text{the zeros of } z^{n+1} - z_0^{n+1} \qquad \text{on } C_\rho$$

and

$$\text{the zeros of } T_{n+1}(z) - T_{n+1}(z_0) \qquad \text{on } \xi_\rho .$$

This is essentially a generalization to these two contours of the Clenshaw–Curtis method [9] for numerical integration on the real line.

Acknowledgments

The author wishes to thank Professor K. O. Geddes of University of Waterloo and Dr. L. W. Longdon and Mr. D. C. Stocks of RMCS Shrivenham for their helpful comments.

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